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Symmetries and Correlation Inequalities for Classical n-Vector Models

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We describe a new class of single spin measures on the n-dimensional sphere S_r^n of radius r ($n \leqslant 4$) for which Lebowitz-type [J. Lebowitz, *J. Stat. Phys.* **16**:463 (1977)] inequalities hold. This is achieved by an appropriate parametrization of S_r^n. The above class includes the uniform measures on $\{x \in \mathbb{R}^n : \rho \leqslant |x| \leqslant r\}$ for any $0 \leqslant \rho \leqslant r$. The second topic of this paper is an abstract formulation of the first Griffiths inequality [R. B. Griffiths, *J. Math. Phys.* **8**:478 (1967)] and the underlying symmetry property.

KEY WORDS: Correlation inequalities; n-vector model; symmetries.

1. INTRODUCTION

Correlation inequalities are a useful tool in proving rigorous results in statistical mechanics. Griffiths[14] has been the first who observed this fact in 1967. Meanwhile his inequalities have been generalized step by step in various directions (see the references, especially the reviews[26,31]). It is of considerable interest to extend these inequalities to the largest class of models in statistical mechanics.

First we present in Section 2 a short analysis of the symmetry argument which is used to prove almost all correlation inequalities. This results in an abstract formulation of the first Griffiths inequality.[14] For which measure spaces $(X, \mathfrak{X}, \omega)$ and sets $\mathfrak{F}$ of functions on X does the inequality

$$0 \leqslant \int_X f(x)\, d\omega(x) \tag{1.1}$$

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hold for all $f \in \mathcal{F}$? We answer this question for measure spaces and sets of functions which are both characterized by simple symmetry properties.

In Section 3 we consider a n-vector model on spheres $S_{r_i}^n$ of radii r_i in $\mathbb{R}^n$ ($n \leqslant 4$) for a finite set of sites Λ, which is defined by the pair (H, ν). The Hamiltonian H is a real-valued function, defined on the configuration space $S = \times_{i \in \Lambda} S_{r_i}^n$ by

$$H(x) = -\sum_{a} \sum_{j=1}^{n} J_a^j \prod_{i \in \Lambda} (x_i^j)^{a(i)} \tag{1.2}$$

for each $x = \{(x_i^1, \ldots, x_i^n)\}_{i \in \Lambda} \in S$, where the first sum ranges over all $a = \{a(i)\}_{i \in \Lambda}$, $a(i) \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and only a finite number of the coupling constants J_a^j is not zero. Finally $\nu = \{\nu_i\}_{i \in \Lambda}$ is the collection of single spin measures on $S_{r_i}^n$. On the configuration space we consider the Gibbs measure

$$d\mu(x) = Z^{-1} \exp[-H(x)] \prod_{i \in \Lambda} d\nu_i(x_i) \tag{1.3}$$

where Z is the partition function such that $\int d\mu(x) = 1$. If $f: S \to \mathbb{R}$ is μ-integrable we denote its expectation value with respect to the Gibbs measure μ by $\langle f \rangle$, keeping the n-vector model in mind. Define for each $a = \{a(i)\}_{i \in \Lambda}$ and $j = 1, \ldots, 4$ a function $X_a^j: S \to \mathbb{R}$ by $X_a^j(x) = \prod_{i \in \Lambda} (x_i^j)^{a(i)}$ if $j \leqslant n$ and $X_a^j \equiv 0$ if $j > n$. Furthermore consider for each finite set $A \subset \times_{i \in \Lambda} \mathbb{N}_0$ and $\epsilon = \{\epsilon(a)\}_a$, $\epsilon(a) \in \{-1, 0, 1\}$ the associated functions $Y_A^\epsilon = \prod_{a \in A} [X_a^1 + \epsilon(a) X_a^3]$, $Z_A^\epsilon = \prod_{a \in A} [X_a^2 + \epsilon(a) X_a^4]$. For which pairs of n-vector models $(H, \nu), (H', \nu)$ with $n \leqslant 4$ does the following type of inequalities hold

$$\langle Y_A^\epsilon Y_B^\tau \rangle - \langle Y_A^\epsilon Y_B^\tau \rangle' \geqslant |\langle Y_A^\epsilon \rangle \langle Y_B^\tau \rangle' - \langle Y_A^\epsilon \rangle' \langle Y_B^\tau \rangle| ? \tag{1.4}$$

We answer this question for the usual conditions on the coupling constants of H and H' and a new class of single spin measures ν_i on $S_{r_i}^n$ including the uniform measure on the set $\{x \in \mathbb{R}^n : \rho_i \leqslant |x| \leqslant r_i\}$ for any $0 \leqslant \rho_i \leqslant r_i$. The description of this class is based on a new parametrization of $S_{r_i}^4$, which couples two plane rotators (i.e., 2-vector models). This coupling is similar to but different from that of Dunlop.[4] The proof of our result mainly follows the well-known pattern.[4,26] The inequality (1.4) can be obtained in the cases $n < 4$ from the case $n = 4$ by choosing special single spin measures among the above class of $n = 4$. For $n > 4$ we can only prove the inequality (1.4) in the case where $J_a^j = 0$ for all $j > 5$ and when ν_i is for example the uniform measure on $S_{r_i}^n$, since integrating the last $n - 4$ variables at each site gives a measure belonging to the above class of $S_{r_i}^4$. To attain a certain completeness we furthermore show that the parametrization used by Dunlop[4] allows us to extend the measure considered by Bricmont[1] in the case $n = 2$ to the case $n \leqslant 4$.

2. ABSTRACT FORMULATION OF THE FIRST GRIFFITHS INEQUALITY

Let $(X, \mathfrak{X})$ be a measure space, and L a countable set. Consider a partition $\mathcal{P} = \mathcal{P}(\{n(i)\}_{i\in L})$ of X into measurable sets $X_i[\sigma^i] \subset X$ with $\sigma^i = (\sigma_1, \ldots, \sigma_{n(i)}) \in \{-1, +1\}^{n(i)}$ for some sequence $\{n(i)\}_{i\in L}$, $n(i) \in \mathbb{N}_0$. Moreover, we have bijective measurable maps $T_i[\sigma^i]$ on X mapping $X_i[\mathbf{1}] = X_i[1, \ldots, 1]$ onto $X_i[\sigma^i]$. Let ω be a measure on $(X, \mathfrak{X})$ which is invariant under each $T_i[\sigma^i]$, and f a measurable function on X satisfying for each $i \in L$ the following two properties:

(1) $0 \leqslant f(x)$ for each $x \in X_i[\mathbf{1}]$.

(2) There exists $(c_1(i), \ldots, c_{n(i)}(i)) \in \{0, 1\}^{n(i)}$ such that for each $x \in X_i[\mathbf{1}]$ and $\sigma^i \in \{-1, +1\}^{n(i)}$ we have

$$f(T_i[\sigma^i](x)) = f(x) \cdot \prod_{j=1}^{n(i)} (\sigma_j)^{c_j(i)}$$

Basic Lemma. Under the above assumptions we have

$$0 \leqslant \int_X f(x)\, d\omega(x) \tag{2.1}$$

Proof. A direct calculation yields

$$\int_X f(x)\, d\omega(x) = \sum_{i\in L} \sum_{\sigma^i \in \{-1,1\}^{n(i)}} \int_{X_i[\sigma^i]} f(x)\, d\omega(x)$$

$$= \sum_{i\in L} \int_{X_i[\mathbf{1}]} \left[f(x) \sum_{\sigma^i \in \{-1,1\}^{n(i)}} \prod_{j=1}^{n(i)} (\sigma_j)^{c_j(i)} \right] d\omega(x) \geqslant 0. \quad \blacksquare$$

Remarks

(1) The above notation allows us to build product spaces $X^1 \times X^2$, and one can consider products of functions $f^1(x^1) f^2(x^2)$, where f^k is compatible with the symmetry structure of X^k, $k = 1, 2$. Instead of one function f we can consider a convergent series of such functions with positive coefficients.

(2) To illustrate the above Basic Lemma we give one example[24]:

$$0 \leqslant \int_0^{2\pi} \int_0^{2\pi} F(\theta, \hat{\theta})\, d\omega(\theta, \hat{\theta}) \tag{2.2}$$

where ω is invariant under $(\theta, \hat{\theta}) \to (\hat{\theta}, \theta)$, $(\theta, \hat{\theta}) \to (\pi - \theta, \pi - \hat{\theta})$ and $(\theta, \hat{\theta}) \to (2\pi - \theta, 2\pi - \hat{\theta})$, and $F(\theta, \hat{\theta})$ is any product of the following terms: $\cos\theta \pm \cos\hat{\theta}$, $\sin\hat{\theta} \pm \sin\theta$, $1 \pm \cos\theta \cos\hat{\theta}$, $1 \pm \sin\theta \sin\hat{\theta}$ (see Fig. 1).

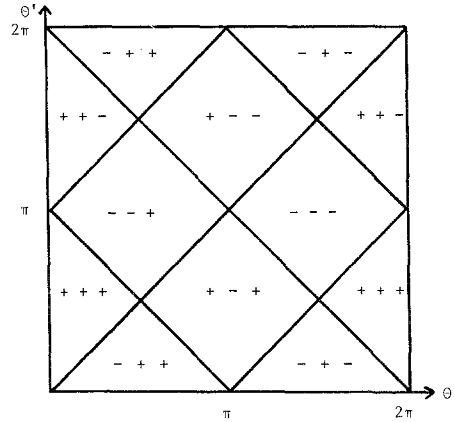

Fig. 1. Partition of $[0,2\pi[\times[0,2\pi[$.

3. CORRELATION INEQUALITIES FOR n-VECTOR MODELS

The result of this section is based on the following parametrization of $x_i=(x_i^1,x_i^2,x_i^3,x_i^4)\in S_{r_i}^4$:

$$\begin{aligned} x_i^1&=r_i\cos\theta_i^1\cos\varphi_i^1, \qquad & x_i^2&=r_i\sin\theta_i^1\cos\theta_i^2\cos\varphi_i^2\\ x_i^3&=r_i\cos\theta_i^1\sin\varphi_i^1, \qquad & x_i^4&=r_i\sin\theta_i^1\cos\theta_i^2\sin\varphi_i^2 \end{aligned}\tag{3.1}$$

with $\theta_i^1,\theta_i^2\in[0,\pi/2]$, $\varphi_i^1,\varphi_i^1\in[0,2\pi[$, $i\in\Lambda$. The single spin measure of the four-vector model will be of the form

$$d\nu_i(x_i)=\exp\left[-\sum_{j=1}^{\infty}a_i^j\left(r_i^2-|x_i|^2\right)^{j/2}\right]\omega_i(\theta_i^1,\theta_i^2)\,d\lambda_i^1(\varphi_i^1)\,d\lambda_i^2(\varphi_i^2)\tag{3.2}$$

where $a_i^j\geqslant 0$ such that the convergence of the series is assured, the product measure $d\omega_i(\theta_i^1,\theta_i^2)\,d\omega_i(\hat\theta_i^1,\hat\theta_i^2)$ is invariant under the exchange of $\theta_i^1\leftrightarrow\hat\theta_i^1$ and $\theta_i^2\leftrightarrow\hat\theta_i^2$, and the measures λ_i^1,λ_i^2 are either the uniform measures on $[0,2\pi[$ or have equal mass in each point $2\pi j/k$, $0\leqslant j\leqslant k-1$ for some $k\in\mathbb{N}$.

Furthermore, we consider on $S_{r_i}^n$, $n\leqslant 4$ the measure

$$d\nu_i(x_i)=\exp\left[-\sum_{j=2}^{\infty}b_i^j|x_i|^{2j}\right]\chi(x_i\in S_{r_i}^n)\,dx_i\tag{3.3}$$

where $b_i^j\geqslant 0$ such that the convergence of the series is assured.

For this measure we replace in the parametrization (3.1)

$$r_i\cos\theta_i^1=\rho_i\quad\text{and}\quad r_i\sin\theta_i^1\cos\theta_i^2=\tau_i\tag{3.4}$$

which is exactly the form considered by Dunlop.[4]

The pair of Hamiltonians (H, H') on $S_{r_i}^n$, $n \leqslant 4$ is related as follows: For each $a = \{a(i)\}_{i \in \Lambda}$ and $\gamma \in \{-1, 1\}$ we have

$$|\hat{J}_a^1 + \gamma \hat{J}_a^3| \leqslant J_a^1 + \gamma J_a^3, \qquad |J_a^2 + \gamma J_a^4| \leqslant \hat{J}_a^2 + \gamma \hat{J}_a^4 \tag{3.5}$$

and if $\|a\| = \sum_{i \in \Lambda} a(i)$ is odd, then additionally

$$J_a^3 = \hat{J}_a^3 = J_a^4 = \hat{J}_a^4 = 0 \tag{3.6}$$

with the convention $J_a^j = \hat{J}_a^j = 0$ if $n < j$.

Theorem. Consider two n-vector models ($n \leqslant 4$) on $S = \times_{i \in \Lambda} S_{r_i}^n$: (H, ν) and (H', ν). Let the single spin measures ν_i be of the form (3.3) or in the case $n = 4$ of the form (3.2), and let the Hamiltonians be related by (3.5) and (3.6). Then for all functions

$$\epsilon, \tau : \underset{i \in \Lambda}{\times} \mathbb{N}_0 \to \{-1, 0, 1\} \qquad \text{with } \epsilon(a) = \tau(a) = 0 \qquad \text{if } \|a\| \text{ is odd} \tag{3.7}$$

and for all $A, B \subset \times_{i \in \Lambda} \mathbb{N}_0$ we have

$$\langle Y_A^\epsilon Y_B^\tau \rangle - \langle Y_A^\epsilon Y_B^\tau \rangle' \geqslant |\langle Y_A^\epsilon \rangle \langle Y_B^\tau \rangle' - \langle Y_A^\epsilon \rangle' \langle Y_B^\tau \rangle| \tag{3.8}$$

$$\langle Z_A^\epsilon Z_B^\tau \rangle' - \langle Z_A^\epsilon Z_B^\tau \rangle \geqslant |\langle Z_A^\epsilon \rangle \langle Z_B^\tau \rangle' - \langle Z_A^\epsilon \rangle' \langle Z_B^\tau \rangle| \tag{3.9}$$

$$\langle Y_A^\epsilon \rangle \langle Z_B^\tau \rangle' - \langle Y_A^\epsilon \rangle' \langle Z_B^\tau \rangle \geqslant |\langle Y_A^\epsilon Z_B^\tau \rangle - \langle Y_A^\epsilon Z_B^\tau \rangle'| \tag{3.10}$$

If moreover in particular $H = H'$, i.e., $|J_a^3| \leqslant J_a^1, |J_a^4| \leqslant J_a^2$ and (3.6) holds, then we have under the above conditions

$$\langle Y_A^\epsilon Y_B^\tau \rangle \geqslant \langle Y_A^\epsilon \rangle \langle Y_B^\tau \rangle, \qquad \langle Z_A^\epsilon Z_B^\tau \rangle \geqslant \langle Z_A^\epsilon \rangle \langle Z_B^\tau \rangle \tag{3.11}$$

$$\langle Y_A^\epsilon \rangle \langle Z_B^\tau \rangle \geqslant \langle Y_A^\epsilon Z_B^\tau \rangle \tag{3.12}$$

Now suppose that $r \leqslant 1$ and $n = 2$, where the single spin measure is of the form (3.3) or the single spin measure satisfies (3.2) plus $\varphi_i^1, \varphi_i^2 \in \{0, \pi\}$. Let the two Hamiltonians H, H' again fulfill (3.5) and (3.6). Then for all $a, b \in \times_{i \in \Lambda} \mathbb{N}_0$, $j = 1, 2$ we have

$$\langle X_a^1 \rangle - \langle X_a^1 \rangle' \geqslant |\langle X_a^1 X_b^j \rangle \langle X_b^j \rangle' - \langle X_a^1 X_b^j \rangle' \langle X_b^j \rangle| \tag{3.13}$$

Remarks

(3) Specializing the single spin measure ν_i on $S_{r_i}^4$ of the form (3.2) one also gets results for $S_{r_i}^n$, $n < 4$. $\varphi_i^2 \in \{0, \pi\}$ gives the case $n = 3$. A parametrization of $S_{r_i}^2$ can be obtained in two different ways, for which the theorem yields two completely different kinds of inequalities. The restriction $\varphi_i^1, \varphi_i^2 \in \{0, \pi\}$ leads to componentwise inequalities using x_i^1 and x_i^2, and choosing $\varphi_i^2 \in \{0\}$, $\theta_i^2 \in \{\pi/2\}$ one obtains vector-coupling inequalities with x_i^1 and x_i^3 as variables. The 1-vector (Ising) model is covered by $\varphi_i^1 \in \{0, \pi\}$, $\varphi_i^2 \in \{0\}$, $\theta_i^2 \in \{\pi/2\}$. As a calculation of the Jacobi determi-

nant shows, the single spin measures

$$d\nu_i^n(x_i) = \exp\left[- \sum_{j=1}^{\infty} a_i^j \left(r_i^2 - |x_i|^2\right)^{j/2} \right] \chi(x_i \in S_{r_i}^n)\, dx_i, \qquad n \leqslant 4, \qquad a_i^j \geqslant 0 \tag{3.14}$$

are obtained as special case of (3.2). For example we have

$$\left| \frac{\partial(x_i^1, x_i^2, x_i^3, x_i^4)}{\partial(\theta_i^1, \theta_i^2, \varphi_i^1, \varphi_i^2)} \right| = r^4 \sin^3\theta_i^1 \sin\theta_i^2 \cos\theta_i^1 \cos\theta_i^2$$

Special cases of (3.14) are

$$\begin{aligned} d\nu_i^n(x_i) &= \chi\left(\left(r_i^2 - s_i^2\right)^{1/2} < |x_i| \leqslant r_i\right) dx_i \\ &= \lim_{k\to\infty} \exp\left[-\left(\left(r_i^2 - |x_i|^2\right)^{1/2} / s_i\right)^k \right] \cdot \chi(x_i \in S_{r_i}^n)\, dx_i \end{aligned} \tag{3.15}$$

(4) The inequalities of the theorem can be extended to the n-vector model, $n > 4$ [or, respectively, $n > 2$ in the situation (3.13)], if $J_a^j = 0$ for all $j > 4$ ($j > 2$), and if the single spin measure on $S_{r_i}^n$ is for example the uniform measure. Since if we keep the first four- (two-) component vector x_i fixed, integration over the remaining $k = n - 4$ ($k = n - 2$) components gives the contribution $(r_i^2 - |x_i|^2)^{k/2} = (r_i \sin\theta_i^1 \sin\theta_i^2)^k$, which can be absorbed in the measure ω_i. Moreover, if $y_i = (y_i^1, \ldots, y_i^{n-4}) \in S_{r_i}^{n-4}$, then we obtain a parametrization of $S_{r_i}^n$ by $x_i^j = r_i \sin\theta_i^1 \sin\theta_i^2\, y_i^{j-4}, n \geqslant j > 4$ in addition to (3.1).

4. PROOF OF THE THEOREM

Each of the inequalities (3.8)–(3.13) can be transformed, using duplicated variables, into the form

$$0 \leqslant \int_{S\times S} G(x,\hat{x}) \exp\left[-H(x) - H'(\hat{x}) \right] \prod_{i\in\Lambda} d\nu_i(x_i)\, d\nu_i(\hat{x}_i) \tag{4.1}$$

For example we obtain the inequalities (3.10) or (3.13) if we choose, respectively,

$$G_\pm(x,\hat{x}) = \left[Y_A^\epsilon(x) \pm Y_A^\epsilon(\hat{x}) \right] \left[Z_B^\tau(\hat{x}) \mp Z_B^\tau(x) \right] \tag{4.2}$$

or

$$G_\pm(x,\hat{x}) = \left[X_a^1(x) - X_a^1(\hat{x}) \right] \left[1 \mp X_b^j(x) X_b^j(\hat{x}) \right] \tag{4.3}$$

Let us first look at (4.2). Introduce the parametrization given by (3.1) and

(3.2) and iteratively apply the identities

$$x_i x_j + \hat{x}_i \hat{x}_j = \tfrac{1}{2}\left[(x_i + \hat{x}_i)(x_j + \hat{x}_j) + (x_i - \hat{x}_i)(x_j - \hat{x}_j)\right] \tag{4.4}$$

$$\cos\varphi_i \cos\varphi_j + \sin\varphi_i \sin\varphi_j = \cos(\varphi_i - \varphi_j) \tag{4.5}$$

to express the terms $\exp[-H(x) - H'(\hat{x})]$ and $G_\pm(x,\hat{x})$ of the form (4.2) as a convergent series with positive coefficients in the variables

$$\cos\theta_i^1 \pm \cos\hat{\theta}_i^1, \qquad \sin\hat{\theta}_i^1 \pm \sin\theta_i^1, \qquad \cos\hat{\theta}_i^2 \pm \cos\theta_i^2, \qquad \sin\theta_i^2 \pm \sin\hat{\theta}_i^2 \tag{4.6}$$

and

$$\cos u\phi^1 \pm \cos u\hat{\phi}^1, \qquad \cos u\hat{\phi}^2 \pm \cos u\phi^2 \tag{4.7}$$

where $u\phi^1 = \sum_{i\in\Lambda} u_i \varphi_i^1$, $u_i \in \mathbb{Z}$. Because of the conditions (3.5), (3.6), (3.7) and the expansion of the exponential function this representation is possible. Using again (4.4) we get for each $i \in \Lambda$ the identity

$$\begin{aligned}
&\exp\left[-a_i^j\left(r_i^2 - |x_i|^2\right)^{j/2}\right]\exp\left[-a_i^j\left(r_i^2 - |\hat{x}_i|^2\right)^{j/2}\right] \\
&\quad = \exp\left[-a_i^j r_i^j\left(\sin^j\theta_i^1 \sin^j\theta_i^2 + \sin^j\hat{\theta}_i^1 \sin^j\hat{\theta}_i^2\right)\right] \\
&\quad = \exp\left[-\tfrac{1}{2}a_i^j r_i^j\left(\sin^j\hat{\theta}_i^1 + \sin^j\theta_i^1\right)\left(\sin^j\theta_i^2 + \sin^j\hat{\theta}_i^2\right)\right] \\
&\qquad \times \exp\left[\tfrac{1}{2}a_i^j r_i^j\left(\sin^j\hat{\theta}_i^1 - \sin^j\theta_i^1\right)\left(\sin^j\theta_i^2 - \sin^j\hat{\theta}_i^2\right)\right]
\end{aligned} \tag{4.8}$$

The first factor is symmetric in $\theta_i^1 \leftrightarrow \hat{\theta}_i^1$ and $\theta_i^2 \leftrightarrow \hat{\theta}_i^2$, and can therefore be absorbed in the measure $d\omega_i(\theta_i^1, \theta_i^2)\, d\omega_i(\hat{\theta}_i^1, \hat{\theta}_i^2)$. The second factor can be expanded as a convergent series with positive coefficients in the variables (4.6) since $a_i^j \geqslant 0$. Because of the symmetry $\theta_i^1 \leftrightarrow \hat{\theta}_i^1$, $\theta_i^2 \leftrightarrow \hat{\theta}_i^2$ we can apply the general result of Section 2, and if $F(\theta_i^1, \hat{\theta}_i^1, \theta_i^2, \hat{\theta}_i^2)$ is any product of functions of the form (4.6) then the Basic Lemma yields

$$0 \leqslant \int F(\theta_i^1, \hat{\theta}_i^1, \theta_i^2, \hat{\theta}_i^2)\, d\omega_i(\theta_i^1, \theta_i^2)\, d\omega_i(\hat{\theta}_i^1, \hat{\theta}_i^2) \tag{4.9}$$

On the other hand the Ginibre inequality[12] gives for $j = 1, 2$

$$0 \leqslant \int \prod_u \left(\cos u\phi^j \pm \cos u\hat{\phi}^j\right) \prod_{i\in\Lambda} d\lambda_i^j(\varphi_i^j)\, d\lambda_i^j(\hat{\varphi}_i^j) \tag{4.10}$$

Thus the inequality (3.10) is proven for the single spin measure (3.2). The inequalities (3.8), (3.9), (3.11), (3.12) can be proven along the same lines. In order to prove the inequality (3.13) for the measure (3.2) we take into consideration the restricting conditions $r \leqslant 1$ and $\varphi_i^1, \varphi_i^2 \in \{0, \pi\}$ and modify the above procedure in the following way:

We do not need the identity (4.5). In addition to the functions (4.6) we need $1 \pm \sin\theta_i^j \sin\hat{\theta}_i^j$, $1 \pm \cos\theta_i^j \cos\hat{\theta}_i^j$, $j = 1, 2$. In place of the functions (4.7) we take those functions obtained by replacing $(\theta_i^j, \hat{\theta}_i^j)$ in (4.6) by $(\varphi_i^j, \hat{\varphi}_i^j)$. Finally instead of the Ginibre inequality (4.10) we apply to the variables $(\varphi_i^j, \hat{\varphi}_i^j)$ a symmetry argument similar as for (4.9).

It remains to prove the theorem for the measure (3.3). We use the parametrization (3.4) and proceed in a similar way as above. We do not need the identity (4.8) and instead of (4.9) we use a remark and a lemma of Bricmont,[1] which say

$$0 \leqslant \int F(\rho_i, \hat{\rho}_i, \tau_i, \hat{\tau}_i)\, d\nu_i(\rho_i, \tau_i)\, d\nu_i(\hat{\rho}_i, \hat{\tau}_i) \tag{4.11}$$

where ν_i is of the form (3.3) for $n = 2$, and F is any product of $(\rho_i \pm \hat{\rho}_i)$, $(\hat{\tau}_i \pm \tau_i)$, $(1 \pm \rho_i\hat{\rho}_i)$, $(1 \pm \tau_i\hat{\tau}_i)$. ∎

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